

INCOMPLETE KLOOSTERMAN SUMS AND HOOLEY'S R*-CONJECTURE

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ABSTRACT. We shall prove an estimate for the incomplete Kloosterman sums to prime moduli, which presents a partial approach to Hooley's R*-Conjecture.

1. INTRODUCTION

Let c be a positive integer. For any integers a and b , the classical Kloosterman sum is defined by

$$S(a, b; c) = \sum_{x \bmod c}^* e\left(\frac{ax + b\bar{x}}{c}\right),$$

where $x\bar{x} \equiv 1 \pmod{c}$ and $*$ restricts the summation to run over the primitive elements. The best possible estimate is essentially due to A. Weil [We]

$$(1) \quad |S(a, b; c)| \leq c^{1/2}(a, b, c)^{1/2}\tau(c),$$

where $\tau(c)$ is the divisor function.

It has been a long time since Kloosterman sums begun to play quite an important role in analytic number theory and automorphic forms, and it turns out it is crucial to prove certain estimates for Kloosterman sums, including the individual one and their mean values, however it is usually rather difficult. For instance, in various arithmetical applications, we may encounter the following general sum

$$S(a, b; c, \mathcal{I}) = \sum_{x \in \mathcal{I}}^* e\left(\frac{ax + b\bar{x}}{c}\right),$$

where \mathcal{I} is an interval with length not exceeding c . This is usually referred to the *incomplete* Kloosterman sum if $\#\mathcal{I} < c$.

By completing method or Fourier technique, one can derive from (1) that

$$S(a, b; c, \mathcal{I}) \ll c^{1/2+\varepsilon}(a, b, c)^{1/2}.$$

However, this estimate is trivial unless $\#\mathcal{I} > c^{1/2+\varepsilon}$. On the other hand, we observe the second moment shows that

$$\frac{1}{c} \sum_{a \bmod c} |S(a, b; c, \mathcal{I})|^2 \leq \#\mathcal{I}.$$

Hence it is reasonable to make the following conjecture:

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Conjecture 1. *Let \mathcal{I} be an interval with $\#\mathcal{I} < c$. Then the estimate*

$$S(a, b; c, \mathcal{I}) \ll \#\mathcal{I}^{1/2+\varepsilon} (a, b, c)^{1/2}$$

holds for any $\varepsilon > 0$.

In particular, in his investigation of Brun-Titchmarsh inequality, C. Hooley [Ho] first stated the following conjecture:

Conjecture 2. *Let \mathcal{I} be an interval with $c^{1/4} < \#\mathcal{I} < c$. Then the estimate*

$$S(0, b; c, \mathcal{I}) \ll \#\mathcal{I}^{1/2+\varepsilon} (b, c)^{1/2}$$

holds for any $\varepsilon > 0$.

The two conjectures are usually referred to Hooley's R^* -Conjecture, since the prototype in Conjecture 2 concerns the incomplete Ramanujan sums.

Based on the method on dealing with double exponential sums of Kloosterman type due to A.A. Karatsuba [Ka], M.A. Korolev [Ko] obtained an upper bound for $S(a, b; c, \mathcal{I})$, which yields

$$(2) \quad S(a, b; c, \mathcal{I}) \ll \#\mathcal{I}(\log c)^{-\delta}$$

for certain $\delta > 0$, provided that \mathcal{I} is of the form $[1, x]$ and $x > \exp(\xi \log c (\log \log c)^{-1})$ for certain $\xi > 0$.

The aim of the present paper is to develop a partial approach to Hooley's R^* -Conjecture and, in particular, prove an estimate for the very special case:

Theorem 1. *Let p be an odd prime and $(b, p) = 1$. Then we have*

$$S(a, b; p, \mathcal{I}) \ll \#\mathcal{I}^{1/r} p^{1/2-(3r-1)/4r^2} \log p$$

for each $r \geq 2$, where the implied constant depends only on r .

One can see that Theorem 1 provides a nontrivial estimate for $\#\mathcal{I} > p^{3/8+\varepsilon}$, in which case we have saved a factor of a positive power of p , instead of a logarithm factor as in (2). In fact, the proof of Theorem 1 depends essentially on the mean value of complete Kloosterman sums over short intervals, which is a special case of Theorem 8 in [Xi].

Proposition 1. *Let m be an integer. For any $x \in \mathbb{R}, y > 0$ and $(b, p) = 1$, we have*

$$\sum_{x < a \leq x+y} S(a, b; p) e\left(\frac{ma}{p}\right) \ll y^{1-1/r} p^{1/2+(r+1)/4r^2} \log p$$

for each $r \geq 2$, where the implied constant depends only on r .

Without the loss of generality, we can assume $\mathcal{I} = (M, M + N]$, M is an integer with $(M, p) = 1$ and $1 \leq N \leq M < p$. By the dyadic device, it suffices to show that

$$(3) \quad S(a, b; p, N) := \sum_{N < n \leq 2N}^* e\left(\frac{a(n+M) + b\overline{(n+M)}}{p}\right) \ll N^{1/r} p^{1/2-(3r-1)/4r^2} \log p$$

for each $r \geq 2$. Here $*$ denotes the restriction that the summation runs over n with $(n+M, p) = 1$, and henceforth we shall use $*$ to remind us such implied coprime restriction.

In addition, we would like to mention that some results in arithmetics might be improved by virtue of Theorem 1, however we shall not deal with any of them in this paper.

Notation. As usual, we write $e(z) = e^{2\pi iz}$, $[x]$ denotes the largest integer not exceeding x and $\{x\} = x - [x]$. The Landau symbol $f = O(g)$ and Vinogradov's notation $f \ll g$ are both understood as $|f| \leq cg$ for certain unspecified $c > 0$. $\widehat{f}(\lambda) = \int_{\mathbb{R}} f(x)e(-\lambda x)dx$ denotes the Fourier transform of f .

2. AUXILIARY RESULTS

First, we shall introduce a smooth function $\phi(x)$ which is compactly supported in $[N, 2N]$ with value 1, and its derivatives satisfy

$$\phi^{(\ell)}(x) \ll N^{-\ell}, \quad \ell \geq 0,$$

where the implied constant in \ll depends only on ℓ . From the alternative integrations by parts, we find that

$$(4) \quad \widehat{\phi}(\lambda) \ll (1 + |\lambda|N)^{-A} N$$

for any $A \geq 0$. Note that we shall take A of different values at different occurrences, however we shall always make use of the letter A .

Now we shall transform the incomplete Kloosterman sum into certain mean value of complete Kloosterman sums.

Lemma 1. *With the notation as above, we have*

$$S(a, b; p, N) = \frac{1}{p} \sum_{n \in \mathbb{Z}} \widehat{\phi}\left(\frac{n}{p}\right) e\left(-\frac{Mn}{p}\right) S(n + a, b; p).$$

Proof. By virtue of ϕ , we can write

$$\begin{aligned} S(a, b; p, N) &= \sum_{n \in \mathbb{Z}}^* \phi(n) e\left(\frac{a(n + M) + b\overline{(n + M)}}{p}\right) \\ &= \sum_{m \bmod p}^* e\left(\frac{a(m + M) + b\overline{(m + M)}}{p}\right) \sum_{\substack{n \in \mathbb{Z} \\ n \equiv m \pmod{p}}} \phi(n). \end{aligned}$$

From Poisson summation, we get

$$\begin{aligned} S(a, b; p, N) &= \frac{1}{p} \sum_{m \bmod p}^* e\left(\frac{a(m + M) + b\overline{(m + M)}}{p}\right) \sum_{n \in \mathbb{Z}} \widehat{\phi}\left(\frac{n}{p}\right) e\left(\frac{mn}{p}\right) \\ &= \frac{1}{p} \sum_{n \in \mathbb{Z}} \widehat{\phi}\left(\frac{n}{p}\right) e\left(-\frac{Mn}{p}\right) S(n + a, b; p). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2. *For any integer m with $(m, p) = 1$, we have*

$$\sum_{a \bmod p} S(a, b; p) e\left(\frac{ma}{p}\right) = pe\left(-\frac{b\overline{m}}{p}\right).$$

Lemma 3. *With the notation as above, we have*

$$\frac{\partial}{\partial x} \widehat{\phi}\left(\frac{x}{p}\right) \ll \frac{N^2}{p} \left(1 + \frac{N|x|}{p}\right)^{-A}$$

for any $A \geq 0$.

Proof. First we have

$$\frac{\partial}{\partial x} \widehat{\phi}\left(\frac{x}{p}\right) = \frac{\partial}{\partial x} \int_{\mathbb{R}} \phi(y) e\left(\frac{xy}{p}\right) dy = \frac{2\pi i}{p} \int_{\mathbb{R}} y \phi(y) e\left(\frac{xy}{p}\right) dy.$$

Then from the alternative integrations by parts we can derive this lemma. \square

3. PROOF OF THEOREM 1

According to Lemma 1, we consider the small n 's and large n 's separately, thus we write

$$\begin{aligned} S(a, b; p, N) &= \frac{1}{p} \sum_{0 < |n| \leq H} \widehat{\phi}\left(\frac{n}{p}\right) e\left(-\frac{Mn}{p}\right) S(n + a, b; p) \\ &\quad + \frac{1}{p} \sum_{|n| > H} \widehat{\phi}\left(\frac{n}{p}\right) e\left(-\frac{Mn}{p}\right) S(n + a, b; p) + \frac{\widehat{\phi}(0)}{p} S(a, b; p) \\ &= \Sigma_1 + \Sigma_2 + O(Np^{-1/2}), \end{aligned}$$

say. Here H is a parameter to be chosen later with

$$H > p, \quad p \mid H.$$

We can estimate Σ_2 trivially by

$$\Sigma_2 \ll p^{-1/2} \sum_{|n| > H} \left| \widehat{\phi}\left(\frac{n}{p}\right) \right| \ll Np^{-1/2} \sum_{n > H} \left(1 + \frac{Nn}{p}\right)^{-A}.$$

In view of (4) and taking $A = 2$, we have

$$(5) \quad \Sigma_2 \ll (NH)^{-1} p^{3/2}.$$

Now we turn to investigate Σ_1 . Alternatively, we only need to deal with

$$\Sigma_1^* = \frac{1}{p} \sum_{n \leq H} \widehat{\phi}\left(\frac{n}{p}\right) e\left(-\frac{Mn}{p}\right) S(n + a, b; p).$$

By partial summation, we have

$$\begin{aligned} \Sigma_1^* &= \frac{1}{p} \widehat{\phi}\left(\frac{H}{p}\right) \sum_{n \leq H} S(n + a, b; p) e\left(-\frac{Mn}{p}\right) \\ &\quad - \frac{1}{p} \int_1^H \left(\sum_{n \leq x} S(n + a, b; p) e\left(-\frac{Mn}{p}\right) \right) \frac{\partial}{\partial x} \widehat{\phi}\left(\frac{x}{p}\right) dx. \end{aligned}$$

From Lemma 2 and (4), we find the first term is bounded by $O(1)$ by taking $A = 2$, thus we have

$$\Sigma_1^* = -\frac{1}{p} \int_1^H \left(\sum_{n \leq x} S(n + a, b; p) e\left(-\frac{Mn}{p}\right) \right) \frac{\partial}{\partial x} \widehat{\phi}\left(\frac{x}{p}\right) dx + O(1).$$

Note that

$$\sum_{n \leq x} S(n + a, b; p) e\left(-\frac{Mn}{p}\right) = \sum_{m \bmod p} S(m + a, b; p) e\left(-\frac{Mm}{p}\right) \sum_{\substack{n \leq x \\ n \equiv m \pmod{p}}} 1$$

$$\begin{aligned}
&= p \left[\frac{x}{p} \right] e \left(\frac{b\overline{M}}{p} \right) + \sum_{m \bmod p} S(m+a, b; p) e \left(-\frac{Mm}{p} \right) \sum_{\substack{p[x/p] < n \leq x \\ n \equiv m \pmod{p}}} 1 \\
&= p \left[\frac{x}{p} \right] e \left(\frac{b\overline{M}}{p} \right) + \sum_{p[x/p] < n \leq x} S(n+a, b; p) e \left(-\frac{Mn}{p} \right),
\end{aligned}$$

we have

$$\begin{aligned}
\Sigma_1^* &= -e \left(\frac{b\overline{M}}{p} \right) \int_1^H \left[\frac{x}{p} \right] \frac{\partial}{\partial x} \widehat{\phi} \left(\frac{x}{p} \right) dx \\
&\quad - \frac{1}{p} \int_1^H \left(\sum_{p[x/p] < n \leq x} S(n+a, b; p) e \left(-\frac{Mn}{p} \right) \right) \frac{\partial}{\partial x} \widehat{\phi} \left(\frac{x}{p} \right) dx + O(1).
\end{aligned}$$

In fact,

$$\begin{aligned}
\int_1^H \left[\frac{x}{p} \right] \frac{\partial}{\partial x} \widehat{\phi} \left(\frac{x}{p} \right) dx &= \int_1^H \left[\frac{x}{p} \right] d\widehat{\phi} \left(\frac{x}{p} \right) \\
&= \frac{H}{p} \widehat{\phi} \left(\frac{H}{p} \right) - \int_1^H \widehat{\phi} \left(\frac{x}{p} \right) d \left[\frac{x}{p} \right] \\
&= \frac{H}{p} \widehat{\phi} \left(\frac{H}{p} \right) - \int_{1/p}^{H/p} \widehat{\phi}(x) d[x] \\
&= \frac{H}{p} \widehat{\phi} \left(\frac{H}{p} \right) - \int_1^{H/p} \widehat{\phi}(x) d[x] \\
&= \frac{H}{p} \widehat{\phi} \left(\frac{H}{p} \right) - \sum_{j=1}^{H/p} \widehat{\phi}(j),
\end{aligned}$$

from which and (4), the first term in Σ_1^* is at most $O(1)$, it follows that

$$\Sigma_1^* = -\frac{1}{p} \int_1^H \left(\sum_{p[x/p] < n \leq x} S(n+a, b; p) e \left(-\frac{Mn}{p} \right) \right) \frac{\partial}{\partial x} \widehat{\phi} \left(\frac{x}{p} \right) dx + O(1).$$

Note that

$$x - p \left[\frac{x}{p} \right] = p \left\{ \frac{x}{p} \right\},$$

then from Lemmas 2 and 3 we have

$$\begin{aligned}
\Sigma_1^* &\ll 1 + p^{1/2+(-3r+1)/4r^2} \log p \int_1^H \left\{ \frac{x}{p} \right\}^{1-1/r} \left| \frac{\partial}{\partial x} \widehat{\phi} \left(\frac{x}{p} \right) \right| dx \\
(6) \quad &\ll 1 + N^2 p^{-1/2+(-3r+1)/4r^2} \log p \int_1^H \left\{ \frac{x}{p} \right\}^{1-1/r} \left(1 + \frac{Nx}{p} \right)^{-A} dx
\end{aligned}$$

for any $A \geq 0$.

Now we write

$$\int_1^H = \int_1^{p/N} + \int_{p/N}^p + \int_p^H := \int_1 + \int_2 + \int_3.$$

First, we have

$$(7) \quad \int_1 \ll p^{-1+1/r} \int_1^{p/N} x^{1-1/r} dx \ll N^{-2+1/r} p.$$

Moreover,

$$\int_2 \ll p^{-1+1/r} \left(\frac{p}{N}\right)^A \int_{p/N}^p x^{1-1/r-A} dx.$$

Now we choose $A = A_1$ such that $1 - 1/r - A_1 < -1$, then we have

$$\int_2 \ll N^{1/r} p^{-1},$$

from which and (7) we can obtain that

$$(8) \quad \int_1 + \int_2 \ll N^{-2+1/r} p.$$

Observing that $\{x\}$ is a periodic function of period 1, we have

$$\begin{aligned} \int_3 &\ll \left(\frac{p}{N}\right)^A \int_p^H \left\{\frac{x}{p}\right\}^{1-1/r} x^{-A} dx \\ &= \left(\frac{p}{N}\right)^A \sum_{j=1}^{H/p-1} \int_0^p \left\{\frac{x}{p}\right\}^{1-1/r} (x+jp)^{-A} dx \\ &\ll p^{-1+1/r} N^{-A} \sum_{j=1}^{H/p-1} j^{-A} \int_0^p x^{1-1/r} dx. \end{aligned}$$

Taking $A = A_2 > 1$, the sum over j is bounded, thus we have

$$(9) \quad \int_3 \ll N^{-A_2} p.$$

Substituting the estimates (8) and (9) into (6), we obtain

$$\begin{aligned} \Sigma_1^* &= N^{1/r} p^{1/2-(3r-1)/4r^2} \log p + N^{2+1/r} p^{-3/2-(3r-1)/4r^2} \log p \\ &\quad + N^{2-A_2} p^{1/2-(3r-1)/4r^2} \log p. \end{aligned}$$

Now taking A_2 sufficiently large, we arrive at

$$\Sigma_1^* \ll N^{1/r} p^{1/2-(3r-1)/4r^2} \log p,$$

thus

$$\Sigma_1 \ll N^{1/r} p^{1/2-(3r-1)/4r^2} \log p.$$

Moreover, in view of (4), we can obtain

$$\Sigma_2 \ll 1$$

by taking $H = p^2$. Notice that

$$N^{1/r} p^{1/2-(3r-1)/4r^2} \log p \geq N p^{-1/2}$$

for each $r \geq 1$, hence we finally arrive at

$$S(a, b; p, N) \ll N^{1/r} p^{1/2-(3r-1)/4r^2} \log p.$$

Note that $r = 1$ provides an estimate even weaker than the trivial one, so we can take $r \geq 2$. Hence we establish (3).

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